A Compatible Cylindrical Shell Element for Stiffened Cylindrical Shells in a Mixed Finite Element Formulation †

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ABSTRACT

In this study using the Gâteaux Differential a functional for thin cylindrical shells with geometric and dynamic boundary conditions are presented. Field equations for thin cylindrical shells are valid under Reissner hypothesis and Hooke’s Law. In this study for the solution of the problem, mixed finite element formulation is used applying variational principles to the functional for thin cylindrical shells a four node rectangular cylindrical shell element is generated. At each node displacements, rotations, in-plane axial and shear forces, transverse shear forces, torsional and bending moments are the unknowns (4*13 degrees of freedom). A uniformly varying thickness property is also included into the formulation and the finite element matrix S13, is explicitly obtained.

1. Introduction

Shells are one of the most important structural forms or structural components in engineering applications. In order to solve general shell problems, due to mathematical complexities, numerical methods are generally preferred. Therefore, the development of finite elements has received special attention from researchers; one of the most widely used numerical methods is the finite element method. A large number of references exist for the analysis of thin shells using the finite element method and a detailed literature survey can be found in [1].

Altın and Igoto [2] carried out important research into the formulation of a mixed finite element for thin cylindrical shells using the Hu-Washizu variational principles, but their boundary conditions matrix is not general. Using a Gâteaux differential, Omurtag and Aköz[3] obtained a new functional which was applicable to all boundary and loading conditions. Later, Omurtag and Aköz[4] applied the formulation to thin orthotropic cylindrical shells. But in all of the above studies, rotational deformations and transverse shear force were ignored in the formulation.

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In this study a new functional, including the transverse shear force, is developed using the two-dimensional shell theory and the Kirchhoff-Love hypothesis. In the derivation of the new functional, the Gâteaux differential is used[5]. Once the functional is obtained, the finite element matrix can be evaluated very easily by applying the variational principles. In the classical formulation, forces and moments can be calculated by a back substitution process. While using this original functional, all internal forces and moments can be obtained directly and more accurately; it also saves computer time. This formulation is also more suitable for the stiffened shell problems, since the rotational compatibility conditions can be satisfied, as well as the displacement compatibility condition in the shell and the stiffener[3,6]. The accuracy and versatility of the new S13 shell element were verified by comparing the solutions of the existing test problems in the literature.

2. The Field Equations for Thin Cylindrical Shells

Using the Kirchhoff-Love hypothesis the field equations for thin cylindrical shells are adequately given in various textbooks[7]. The global coordinate axes, the displacements and the positive directions of the distributed external forces and moments are shown in Fig.1. Stress resultant, stress couple and transverse shear force components are shown in Fig. 2. The field equations are written as below.

The equilibrium equations

\[
\begin{align*}
\frac{\partial Q}{\partial s} + \frac{\partial P}{\partial x} + q_x &= 0 \\
\frac{\partial N}{\partial s} + \frac{\partial Q}{\partial x} + \frac{H}{R} + q_s &= 0 \\
\frac{\partial F}{\partial x} + \frac{\partial H}{\partial s} + \frac{N}{R} + q_z &= 0 \\
\frac{\partial K}{\partial x} + \frac{\partial T}{\partial s} - F + m_x &= 0 \\
\frac{\partial M}{\partial s} + \frac{\partial T}{\partial x} - H + m_s &= 0.
\end{align*}
\] (1)

Figure 1. Global coordinate axes and positive directions of (a) distributed external force loadings and (b) displacements and distributed external moment loadings.
Figure 2. (a) Positive directions of the force components. (b) Positive directions of the moment components.

\[ P - B \left( \nu \frac{\partial v}{\partial s} + \nu \frac{\partial u}{\partial x} + W \frac{R}{s} \right) = 0 \]

\[ N - B \left( \frac{\partial v}{\partial s} + \frac{\partial u}{\partial x} \right) W = 0 \]

\[ Q - B \left( \frac{l - \nu}{2} \right) \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial s} \right) = 0 \]

\[ F - C \left( \frac{\partial w}{\partial x} + \Omega_x \right) = 0 \]

\[ H - C \left( \frac{\partial w}{\partial s} + \Omega_s - \frac{\nu}{R} \right) = 0 \]

\[ K - D \left( \frac{\partial \Omega_x}{\partial x} + \nu \frac{\partial \Omega_s}{\partial s} \right) = 0 \]

\[ M - D \left( \frac{\partial \Omega_x}{\partial x} + \frac{\partial \Omega_s}{\partial s} \right) = 0 \]

\[ T - D \left( \frac{1 - \nu}{2} \right) \left( \frac{\partial \Omega_x}{\partial s} + \frac{\partial \Omega_s}{\partial x} \right) = 0. \] (2)

where

\[ B = \frac{Eh}{(l - \nu^2)} \]

\[ D = \frac{Eh^3}{12(l - \nu^2)} \]

\[ C = \frac{5}{6} \frac{Eh}{6(1 + \nu)} \] (3)

are the tensile, flexural and shear rigidities, respectively. The dynamic boundary conditions

\[ - \tau + \ddot{\tau} = 0, \quad \xi - \ddot{\xi} = 0 \] (4)

and the geometric boundary conditions

\[ - \Omega + \dot{\Omega} = 0, \quad \delta - \dot{\delta} = 0 \] (5)

are written in symbolic form in eqns (4) and (5). Quantities with the hat have known values on the boundary. \( \tau, \xi, \delta, \Omega \) are force, moment, displacement and rotation vectors, respectively. Field equations can be written in operator form as
\[ Q = \mathcal{L}Y - f. \] (6)

The matrix form of the operator is given in the Appendix.

3. Functionals for Shells

If the operator in eqn. (6) is a potential operator the equality

\[ < dQ(y, \bar{y}), y^* > = < dQ(y, y^*), \bar{y} > \] (7)

must be satisfied [5]. Where \( dQ(y, \bar{y}) \) is the Gâteaux derivative of \( Q \) and \( < \, dQ(y, \bar{y}), y^* > \) is the inner product of two vectors. Inner products on the boundary are defined as follows:

\[
\begin{align*}
[\tau, \delta] &= [Q, u] + [P, u] + [Q, \nu] + [N, \nu] \\
&\quad + [F, w] + [H, w] \\
[\xi, \Omega] &= [K, \Omega_x] + [T, \Omega_x] + [M, \Omega_s] + [T, \Omega_s].
\end{align*}
\] (8)

It can be shown that the operator for thin shells satisfies eqn(7). In this case, functional corresponding to the field equations is obtained as [5]

\[ \mathcal{L}(y) = \int_0^1 < Q(sy, y), y > \, ds, \] (9)

where \( s \) is a scalar quantity. The functional is obtained as

\[
I_s(y) = -\left[ Q, \frac{\partial u}{\partial s} \right] - \left[ P, \frac{\partial u}{\partial x} \right] - \left[ N, \frac{\partial \nu}{\partial s} \right] - \left[ Q, \frac{\partial \nu}{\partial x} \right] \\
-\frac{1}{R} [\nu, H] - \left[ \frac{\partial w}{\partial x}, F \right] - \left[ \frac{\partial w}{\partial s}, H \right] - \frac{1}{R} [N, W] \\
+ \left[ \frac{\partial K}{\partial X}, \Omega_x \right] + \left[ \frac{\partial T}{\partial s}, \Omega_x \right] - [F, \Omega_x] \\
+ \left[ \frac{\partial M}{\partial s}, \Omega_s \right] + \left[ \frac{\partial T}{\partial x}, \Omega_s \right] - [H, \Omega_s] + [q_x u] \\
+[q_s, \nu] + [q_z, w] + [m_x, \Omega_x] \\
+[m_s, \Omega_s] + \frac{1}{2C} \{[F, F] + [H, H]\} \\
+ \frac{1}{2B(1-\nu^2)} \{[P, P] + [N, N] - 2\nu[P, N]\} \\
+ \frac{1}{B(1-\nu)} [Q, Q] + \frac{1}{D(1-\nu)} [T, T]
\]
\begin{align}
+ \frac{1}{2D(1-\nu^2)} \{ [K, M] + [M, M] - 2\nu[K, M]\} \\
+[u, Q] + [u, \tilde{P}] + [\nu, \tilde{Q}] + [\nu, \tilde{N}] \\
+[w, \tilde{F}] + [w, \tilde{H}] - [\Omega_x, (K - \bar{K})] \\
-\{\Omega_x, (T - \bar{T})\} - [\Omega_s, (M - \bar{M})] \\
-\{\Omega_s, (T - \bar{T})\}\sigma + \{- [K, \dot{\Omega}_x] - [T, \dot{\Omega}_x] \\
- [M, \dot{\Omega}_s] - [T, \Omega_s] + [P, (u - \bar{u})] \\
+ [N, (\nu - \bar{\nu})] + [Q, (u - \bar{u})] + [Q, (\nu - \bar{\nu})] \\
+ [F, (w - \bar{w})] + [H, (w - \bar{w})]\} c. \tag{10}
\end{align}

The braces with the \( \sigma \) index and \( \epsilon \) index are valid on the boundary where the dynamic boundary conditions and the geometric boundary conditions are prescribed, respectively.

4. Isoparametric Finite Element Formulation for Cylindrical Shells

Let \( u, \nu, w \) be the displacement components along the \( X \) axis in the circumferential (\( S \)) and normal (\( Z \)) directions; \( \Omega_x \) and \( \Omega_s \) being the rotations of the cross-sections normal to \( X \) and \( S \) in the cylindrical shell. \( P, N, Q \) are in-plane axial and shear forces, \( F, H \) are transverse shear forces, and \( K, M, T \) are the bending and torsional moment components. They are the nodal unknowns of the newly generated finite element and are expressed by shape function \( \Psi_i \) in the element. For example, \( u = \sum u_i \Psi_i \) where \( u_i \) are the nodal values and \( i = 1, \cdots, n (n = \text{number of nodes of the element}) \). Details can be found in [3].

The formulation used in the previous section can be very easily extended to develop a \( C^0 \) rectangular cylindrical shell element. For this purpose a rectangular isoparametric element is used. The master element is shown in Fig. 3.

![Figure 3. Master element.](image)

By variational principles, from eqn (10), one obtains the finite element matrix \( [k]_s \) as follows:
\[
[k]_s = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma[k_1] & -\alpha[k_1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma[k_1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu[k_1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\varphi[k_1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\varphi[k_1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\delta[k_1] & -\beta[k_1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\lambda[k_1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]  
(11)

in which

\[
\begin{align*}
\varphi &= \frac{1}{C}, \quad \gamma = \frac{1}{Eh}, \quad \delta = \frac{12}{(Eh^3)} \\
\alpha &= \nu \gamma, \quad \beta = \nu \delta, \quad \mu = \frac{2(1 + \nu)}{(Eh)} \\
\lambda &= \frac{24(1 + \nu)}{(Eh^3)}. \quad (12)
\end{align*}
\]

The submatrices \([k_1],[k_2]\) and \([k_3]\) are given in the Appendix. The boundary conditions matrix is written as follows:

\[
[k]_{BC} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]  
(13)

The submatrices \([s_1]\) and \([s_2]\) are given in the Appendix. The details of the mathematical manipulations can be found in [1,3,4].

Finally, the cylindrical shell finite element matrix for uniform thickness is obtained as
\[ [k]_s = [k]_s + [k]_{BC}. \] (14)

5. Numerical Examples

The feasibility and accuracy of the proposed finite element formulation of thin cylindrical shells presented in this research are demonstrated by the following two very popular cylindrical shell roof problems. They have been used to test finite element analyses by other authors. Figure 4 illustrates the geometry of Examples 1 and 2. A uniform cylindrical shell with diaphragm end supports is subjected to a gravity load of \( q = 4.393kN/m^2 \). Material properties and dimensions are

\[
E = 2.106 \times 10^7 kN/m^2, \quad \nu = 0.3 \\
l = 15.24m, \quad \theta = 40^\circ \\
R = 7.62m, \quad h = 0.0762m.
\]

Due to symmetry only one quarter of the shell is analysed. For the sake of simplicity \( g \simeq 10m/sec^2 \) is used.

*Table 1. Comparison of some displacement results at the typical points of Example 1 (Fig.4). Numbers in parentheses give the meshes*

<table>
<thead>
<tr>
<th>Reference</th>
<th>( w_g ) (cm)</th>
<th>( w_F ) (cm)</th>
<th>( u_P ) (cm)</th>
<th>( \nu_G ) (cm)</th>
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<td>-</td>
<td>-</td>
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<td>(4x5)</td>
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<tr>
<td>[9]</td>
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<td>1.37</td>
<td>-</td>
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<td>(5x6)</td>
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<tr>
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<td>-0.38</td>
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<tr>
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<tr>
<td>[8]</td>
<td>-9.39</td>
<td>1.40</td>
<td>-0.30</td>
<td>2.23</td>
</tr>
<tr>
<td></td>
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Table 2. Comparison of some force and bending moment results at the typical points of Example 1 (Fig.4). Numbers in parentheses give meshes

<table>
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<tr>
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<th>$M_F$ (kN)</th>
<th>$K_F$ (kN)</th>
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<td>(5x5)</td>
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</tr>
<tr>
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<td>-10.7</td>
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<td>(16x22)</td>
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<td>(4x4)</td>
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<tr>
<td>[8]</td>
<td>-11.4</td>
<td>10.0</td>
<td>4.2</td>
</tr>
</tbody>
</table>

**Example 1**

The sides AC and BD are free in Fig.4. Table 1 shows displacements and Table 2 force and moment results at some typical points and these are also compared with the literature results.

*Figure 5. Variation of displacements along some boundaries of Example 2.*
Example 2

The sides AC and BC are hinged in Fig. 4. Variation of displacement force and moment results along some boundaries are given in Figs 5-7.

Figure 6. Variation of force components along some boundaries of Example 2.

Figure 7. Variation of moment components along some boundaries of Example 2.

6. Conclusions

A new function is presented for thin cylindrical shells including transverse shear force components with other force and moment components as well as translational and rotational deformations, using the two-dimensional shell theory and the Kirchhoff-Love hypothesis. The functional is suitable for a mixed finite element construction. An isoparametric $C^0$ class rectangular element is presented. This mixed finite element formulation is especially compatible with other mixed finite element formulated space bars, since three translational deformations and two rotational deformations are defined as unknowns together with the force and the moment components. Besides distributed external loading (i.e. gravity load) and alternative loading of distributed moment is possible with this formulation.
In the classical finite element formulation, a coefficient matrix calculation and a backsubstitution process are unavoidable; of course, additional operations do consume computer time. In this paper a specialized formulation is presented which overcomes both of these time-consuming procedures.

Full agreement is obtained between the presented formulation and the exact solution of the examples. Comparisons with the classical finite element methods show the versatility and the accuracy of this new formulation.

**NOTATION**

- $\mathcal{L}$ \hspace{1em} coefficient matrix
- $l(y)$ \hspace{1em} functional
- $E, \nu$ \hspace{1em} Young's modulus and Poisson's ratio
- $X, S, Z$ \hspace{1em} global coordinate axes
- $\xi, \eta$ \hspace{1em} coordinate axes of master element
- $[\cdot, \cdot]$ \hspace{1em} inner product of two vectors
- $[\cdot, \cdot]_{ab}$ \hspace{1em} dynamic boundary conditions
- $[\cdot, \cdot]_{bc}$ \hspace{1em} geometric boundary conditions
- $R, h$ \hspace{1em} radius and thickness of cylindrical shell
- $\Omega, d$ \hspace{1em} rotation and displacement vector
- $u, v, w$ \hspace{1em} tangential and normal displacements
- $P, N, Q$ \hspace{1em} stress resultants, in-plane shear force
- $K, M, T$ \hspace{1em} stress couple resultants, torsional moment
- $F, H$ \hspace{1em} transverse shear force components
- $\Omega_x, \Omega_s$ \hspace{1em} rotations for sections $x=$ constant and $s=$ constant, respectively
- $B, D, C$ \hspace{1em} tensile, flexural and shear rigidities
- $q_x, q_s, q_z$ \hspace{1em} distributed load acting along $X, S, Z$ axes
- $m_x, m_s$ \hspace{1em} distributed moments with respect to $X, S$ axes
- $\Psi_i$ \hspace{1em} shape functions, $i = 1, \ldots, 4$
- $a, b$ \hspace{1em} half length of the side of a rectangular element
- $[\cdot]$ \hspace{1em} matrix
- $[k_i], [S_i]$ \hspace{1em} submatrices
- $[k]_{BC}$ \hspace{1em} boundary conditions matrix of shell element
- $r, \xi, \delta, \Omega$ \hspace{1em} symbolic form of force, moment, displacement rotation vectors
- $f, \xi, \delta, \Omega$ \hspace{1em} known values on the boundary

**References**


